

RATIONAL CONNECTEDNESS AND ORDER OF NON-DEGENERATE MEROMORPHIC MAPS FROM \mathbb{C}^n

FRÉDÉRIC CAMPANA & JÖRG WINKELMANN

ABSTRACT. We show that an n -dimensional compact Kähler manifold X admitting a non-degenerate meromorphic map $f : \mathbb{C}^n \rightarrow X$ of order $\rho_f < 2$ is rationally connected.

1. INTRODUCTION/SUMMARY

The purpose of this paper is the following result.

Theorem 1.1. *Let X be a compact Kähler manifold of fixed dimension n . Let $f : \mathbb{C}^n \dashrightarrow X$ be a non-degenerate meromorphic map of order $\rho_f < 2$ (see the next section for the definition of ρ_f).*

Then X is rationally connected, hence projective.

This result belongs to a series of similar statements relating the existence and growth of maps from \mathbb{C}^n to algebro-geometric properties of the target space X .

These statements are better expressed by introducing the following invariant $\rho(X)$, which suggests many questions, some of which are raised in the last section:

Definition. *Let X be an n -dimension connected compact complex manifold. Define $\rho(X) := \inf\{\rho_f | f : \mathbb{C}^m \dashrightarrow X, m \geq n\}$, f non-degenerate.*

It is understood that $\rho(X) \in [0, +\infty]$, and that $\rho_f = +\infty$ if there exists no non-degenerate meromorphic map $f : \mathbb{C}^n \rightarrow X$.

The invariant ρ is easily seen to be bimeromorphic, preserved by finite étale covers, and ‘increasing’ (ie: $\rho(X) \geq \rho(Y)$ if there exists a dominant meromorphic map $g : X \rightarrow Y$).

It is obvious that $\rho(X) = 0$ if X is unirational.

Kobayashi and Ochiai proved that the existence of a non-degenerate map from \mathbb{C}^n to a projective manifold X implies that X is not of general type ([10]). Thus $\rho(X) = +\infty$ if X is of general type. It is proved in [4], more generally, that X is ‘special’ if there exists a non-degenerate

Acknowledgement. The second author was supported by the SFB/TR 12 “Symmetries and Universality in mesoscopic systems”.

meromorphic map $f : \mathbb{C}^n \dashrightarrow X$. In particular, X must be special if $\rho(X) < +\infty$.

If X is Kähler and K_X is pseudo-effective of numerical dimension $\nu \in \{0, 1, \dots, n\}$, then $\rho(X) \geq 2(1 - \frac{\nu}{n})^{-1} \geq 2$ ([6]). This implies the previous result of Kobayashi-Ochiai (since X is of general type if and only if $\nu = n$). Using [2], it also implies that if X is projective, then X is uniruled if $\rho(X) < 2$.

If $h^0(X, \text{Sym}^k(\Omega_X^p)) \neq 0$, for some $k, p > 0$, then $\rho(X) \geq 2$ ([12]). The Kähler condition is not required here. If, in addition, X is assumed to be a Kähler surface, then the condition $h^0(X, \text{Sym}^k(\Omega_X^p)) = 0 \forall p, k$ implies that X is Kähler. Thus, if X is a Kähler surface and if $\rho(X) < 2$, then X must be rational ([12]). On the other hand, some Hopf (thus non-rational, non-Kähler) surfaces have $\rho(X) \leq 1$ ([12]).

The present result generalizes these two results from [6], [12], avoiding the deep methods of [6]. The estimate $\rho_f < 2$ is optimal, since for an Abelian variety A we have $\rho_\tau = 2$ for the universal covering map $\tau : \mathbb{C}^g \rightarrow A$.

All these results provide lower bounds for $\rho(X)$ deduced from the geometry of X . Producing upper bounds for $\rho(X)$ (ie: the existence of non degenerate f 's) turns out to be a completely open topic, except in the trivial case of X unirational or a torus.

The case of rationally connected vs unirational manifolds when $n \geq 3$ is of great interest. For example: what is $\rho(X)$ if X is a ‘general’ smooth quartic in \mathbb{P}^4 ? If $\rho(X) > 0$, then X is not unirational. More generally: are there projective (necessarily rationally connected) X such that $\rho(X) \in]0, 2[$? See the last section for some more questions.

2. CHARACTERISTIC FUNCTION AND ORDER OF A NON-DEGENERATE MEROMORPHIC MAP

We start with some preparations. Let X be a compact complex manifold and let $f : \mathbb{C}^n \dashrightarrow X$ be a meromorphic map. f is said to be (differentiably) “non-degenerate” if there is a point $p \in \mathbb{C}^n$ such that f is holomorphic at p and such that $(Df)_p : T_p \mathbb{C}^n \rightarrow T_{f(p)} X$ is surjective.

Let $\alpha := dd^c ||z||^2$ on \mathbb{C}^n , and let ω be a positive $(1, 1)$ -form on X . The characteristic function of f is defined as:

$$T_f(r; \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} (f^* \omega) \wedge \alpha^{n-1}.$$

Here $B_t = \{z \in \mathbb{C}^n : |z| < t\}$. Observe that the integral over B_t is well-defined even if f is only meromorphic, not necessarily holomorphic. (To see this, note that locally ω can be dominated by a sum $\sum_i \alpha_i \wedge \bar{\alpha}_i$ where the α_i are holomorphic 1-forms. The holomorphy of the α_i implies that

$\sum_i \alpha_i \wedge \bar{\alpha}_i$ extends real-analytically through the indeterminacy set of f .)

If ω and $\tilde{\omega}$ are any two positive $(1,1)$ -forms on X , then (by the compactness of X) there are constants $C_1, C_2 > 0$ such that

$$(2.1) \quad C_1 \omega < \tilde{\omega} < C_2 \omega$$

and consequently

$$C_1 T_f(r, \omega) < T_f(r, \tilde{\omega}) < C_2 T_f(r, \omega) \quad \forall r > 1.$$

The “order” ρ_f is defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r, \omega)}{\log r}$$

where ω is a positive $(1,1)$ -form on X . Due to the inequalities 2.1 this number $\rho_f \in \mathbb{R} \cup \{+\infty\}$ is independent of the choice of the positive $(1,1)$ -form ω used in the definition of $T_f(r, \omega)$.

The well-known Crofton’s formula (see e.g. [8]) implies that $T_f(r)$ equals the average of $T_{f|L}(r)$ taken over all complex lines $L \subset \mathbb{C}^n$. This permits to extend many properties of characteristic functions from the case of entire curves to the higher-dimensional domains.

For instance, let $\tau : X' \rightarrow X$ be a bimeromorphic holomorphic map between compact complex manifolds. Then $\tilde{f} = \tau^{-1} \circ f : \mathbb{C}^n \rightarrow X'$ is again a non-degenerate meromorphic map, and $\rho_f = \rho_{\tilde{f}}$.

Now let $\alpha : X \rightarrow Y$ be a dominant holomorphic map. Then it follows directly from the definitions that $T_{\alpha \circ f}(r) \leq T_f(r)$ (if $\omega_X \geq \alpha^*(\omega_Y)$) and consequently $\rho_{\alpha \circ f} \leq \rho_f$.

The order is easily seen to behave nicely with respect to products. Let $X = X_1 \times X_2$, let ω_i be positive $(1,1)$ -forms in X_i and let $f_i : \mathbb{C}^{n_i} \rightarrow X_i$ be non-degenerate meromorphic maps. Define $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$, $n = n_1 + n_2$ and $f : \mathbb{C}^n \rightarrow X$ as $f(v_1, v_2) = (f_1(v_1), f_2(v_2))$.

Then $T_f(r) = T_{f_1}(r) + T_{f_2}(r)$. This implies $\rho_f = \max\{\rho_{f_1}, \rho_{f_2}\}$, because for any $t_1, t_2 > 0$ we have

$$\max_i \log t_i \leq \log(t_1 + t_2) \leq \log(2 \max_i t_i) = \log 2 + \log \max_i t_i.$$

3. RATIONAL CONNECTEDNESS

As usual, a compact complex manifold X (usually Kähler) is called “rationally connected” (short: RC) if every two points can be linked by a (possibly singular) irreducible rational curve. (Equivalently in the Kähler case: can be linked by a chain of rational curves). Unirational manifolds are RC. There are unirational threefolds which are not rational, e.g. smooth cubic hypersurfaces in \mathbb{P}_4 . However, it is not known whether there exist RC manifolds which are not unirational, although

it is expected that these should exist (the case of ‘general’ quartics in \mathbb{P}_4 being one of the first open cases) .

Rationally connected compact Kähler manifolds are projective (since $h^{2,0} = 0$, using Kodaira’s criterion).

Let X be a compact connected Kähler manifold. Then there exists an “almost holomorphic” rational dominant map $\rho : X \dashrightarrow Y$ (called the “RC-reduction”, or “rational quotient” [3], or the “MRC-fibration” [11] if X is projective), such that the fibers are RC, and maximum with this property.

When X is projective, it is known (by [9]) that the base Y is not uniruled. In fact, [9] shows that if $f : X \dashrightarrow Y$ is a surjective meromorphic map with fibres and base Y which are both RC, then X is RC if it is compact Kähler (remark first that $h^{2,0}(X) = 0$, and that X is thus projective). The base Y of the RC-reduction $\rho : X \dashrightarrow Y$ is not uniruled also when X is compact Kähler. Let indeed $r : Y \dashrightarrow Z$ be the RC-reduction of Y . The fibres of $r \circ \rho : X \dashrightarrow Z$ are thus RC, and so $Y = Z$, which means that Y is not uniruled.

Due to [2] a projective manifold is uniruled if and only if K_X is not pseudoeffective. Based on this, another recent criterion is the following (see [5]):

Let X be a compact Kähler manifold. Then X is rationally connected if and only if there is no pseudoeffective invertible subsheaf $F \subset \Omega_X^p$. (for some $p \in \mathbb{N}$).

The proof consists in observing that X is not RC precisely if its RC-reduction $\rho : X \dashrightarrow Y$ has $\dim(Y) = p > 0$. Define then $F := \rho^*(K_Y)$, which is pseudo-effective since Y is not uniruled.

It is conjectured that a compact Kähler (or equivalently: projective) manifold X is RC if (and only if) there is no \mathbb{Q} -effective (instead of pseudo-effective) invertible subsheaf $F \subset \Omega_X^p$ (for some $p \in \mathbb{N}$). By means of the RC-reduction as above, this conjecture is equivalent to the ‘non-vanishing conjecture’, claiming that if K_X is pseudo-effective, it is \mathbb{Q} -effective.

4. PSEUDOEFFECTIVE LINE BUNDLES

A singular hermitian metric h on a complex line bundle is given in the form $|\cdot|_h = e^{-\phi} |\cdot|_s$ where s is the standard metric for some local holomorphic trivialization and ϕ is a L_{loc}^1 -function.

The L_{loc}^1 -condition ensures that the curvature $\Theta = dd^c \log(e^{-\phi}) = -dd^c \phi$ makes sense (in the sense of currents) and represents the Chern class of the line bundle.

A line bundle L on a compact complex manifold is called “pseudo-effective” iff there is a singular hermitian metric h with semipositive curvature $\Theta_h \geq 0$. This condition means that the metric is locally given via a weight function $e^{-\phi}$ with ϕ plurisubharmonic. If s is a holomorphic section in F , this implies that $-\log \|s\|_h$ is plurisubharmonic.

5. MEASURING THE DERIVATIVE

Let V, W be complex vector spaces equipped with hermitian inner products. The norm $\|F\|$ of a complex linear map $F : V \rightarrow W$ is defined by: $\|F\|^2 = \text{trace}(F^* \circ F)$ where F^* denotes the adjoint of F . If A is the matrix describing F with respect to orthonormal bases on V and W , then

$$\|F\|^2 = \sum_{i,j} |A_{ij}|^2.$$

We continue with a local observation. Let U, V be open subsets in \mathbb{C}^n , let $\alpha = dd^c \|z\|^2 = \sum_j i \cdot dz_j \wedge d\bar{z}_j$ be the Kähler form for the euclidean metric and let $f : U \rightarrow V$ be a holomorphic map.

Then

$$f^* \alpha \wedge \alpha^{n-1} = \frac{1}{n} \|Df\|^2 \alpha^n$$

where

$$\|Df\|^2 = \text{trace}((Df)^* \circ (Df)) = \sum_{j,k} \left| \frac{\partial f_j}{\partial z_k} \right|^2$$

Proposition 5.1. *Let $U \subset \mathbb{C}^n$ be an open subset, let X be a compact Kähler manifold equipped with a hermitian metric h and a positive $(1,1)$ -form ω and let $f : U \rightarrow X$ be a holomorphic map.*

Then there is a constant $C > 0$ such that

$$f^* \omega \wedge \alpha^{n-1} \geq C \|Df\|^2 \alpha^n.$$

Here $\|Df\|$ is calculated with respect to h .

Proof. We cover X with finitely many open subsets V_k such that each V_k admits an embedding $j_k : V_k \rightarrow \mathbb{C}^n$ and each V_k contains a relatively compact open subset $W_k \subset V_k$ such that $X = \cup_k W_k$. Let h_k denote the hermitian metric on V_k induced by the euclidean metric via its embedding in \mathbb{C}^n . We choose $C_1 > 0$ such that $h_k \leq C_1 h$ everywhere on each W_k and $C_2 > 0$ such that $\omega \geq C_2 j_k^* \alpha$ on each W_k . Then the claim (with $C = C_1 C_2$) follows from the preceding local observation. \square

6. THE RESULT

We show:

Theorem 6.1. *Let X be a compact Kähler manifold. Let $f : \mathbb{C}^n \dashrightarrow X$ be a non-degenerate meromorphic map of order $\rho_f < 2$.*

Then X is projective and rationally connected.

Proof. Assume by contradiction that X is not rationally connected. There is then a pseudoeffective invertible subsheaf $\mathcal{F} \subset \Omega_X^p$ (for some $p \in \mathbb{N}$). We fix a hermitian metric h on X . The hermitian metric on T_X induces a hermitian metric on Ω_X^p , by abuse of language also denoted by h . The injection of sheaves $\mathcal{F} \hookrightarrow \Omega_X^p$ corresponds to a non-zero vector bundle homomorphism $\xi_0 : F \rightarrow \Omega_X^p$, where F is a pseudoeffective line bundle. Let g denote a smooth hermitian metric on F with $g \leq h|_F$ and let g_0 denote a singular hermitian metric on F such that $\Theta_{g_0} \geq 0$, i.e, with positive curvature current. Then there is an upper semicontinuous function $\phi \rightarrow \mathbb{R}$ such that $g_0 = e^{-\phi}g$. Since ϕ is upper semicontinuous, and X is compact, it is bounded from above: $M := \sup_{x \in X} \phi(x) < \infty$.

The meromorphic map $f : \mathbb{C}^n \rightarrow X$ and the vector bundle homomorphism $\xi_0 : F \rightarrow \Omega_X^p$ induce vector bundle homomorphisms

$$f^*F \xrightarrow{\xi} f^*\Omega_X^p \xrightarrow{Df^*} \Omega_{\mathbb{C}^n}^p.$$

on $\mathbb{C}^n \setminus I(f)$ (with $I(f)$ denoting the indeterminacy set of the meromorphic map f .) We are interested in a lower bound for $\|Df\| = \|(Df)^*\|$, calculated with respect to the metric induced by h resp. the euclidean metric on $f^*\Omega_X^p$ resp. $\Omega_{\mathbb{C}^n}^p$. Let $\beta := (Df^*) \circ \xi$. Since we assumed $g \leq h|_F$, we have $\|\beta\|_g \leq \|Df\|$ where $\|\beta\|_g \leq \|Df\|$ denotes the norm with respect to g on F and the euclidean metric on $\Omega_{\mathbb{C}^n}^p$. Let $\|\beta\|_{g_0}$ denote the norm with respect to g_0 on F .

By using the standard trivialization of $\Omega_{\mathbb{C}^n}^p$, the bundle morphism $\beta : f^*F \rightarrow \Omega_{\mathbb{C}^n}^p$ corresponds to a vector valued section on the dual bundle f^*F^* . Now $\Theta_{g_0} \geq 0$ implies that $\log \|s\|_{g_0}$ is plurisubharmonic for every holomorphic section s in f^*F^* .

Hence $\log \|\beta\|_{g_0}$ is a plurisubharmonic function on $\mathbb{C}^n \setminus I(f)$ where $I(f)$ denotes the indeterminacy set of the meromorphic map f . Plurisubharmonic functions extend through closed analytic subsets of codimension at least two. Hence $\log \|\beta\|_{g_0}$ extends to a plurisubharmonic function ζ_0 defined on the whole \mathbb{C}^n .

We observe that $\|\beta\|_{g_0} = e^\phi \|\beta\|_g$, because $g_0 = e^{-\phi}g$.

By the definition of the constant M , we have $\|\beta\|_{g_0} \leq e^M \|\beta\|_g$, implying

$$\zeta_0 - M \leq \log \|\beta\|_g \leq \log \|Df\|$$

Define $\zeta = \exp(\zeta_0 - M)$. The plurisubharmonicity of ζ_0 implies the plurisubharmonicity of ζ . Thus we obtain the existence of a plurisubharmonic function ζ on \mathbb{C}^n such that $\|Df\| \geq \zeta$.

Using proposition 5.1 we may deduce that

$$f^* \omega \wedge \alpha^{n-1} \geq \zeta \alpha^n.$$

It follows that

$$T_f(r; \omega) \geq \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \zeta \alpha^n$$

Since moving the origin, i.e., replacing f by $f \circ \tau$ where τ denotes a translation, does not affect the order ρ_f , we may assume that $c = \zeta(0, \dots, 0) > 0$. Using the sub-mean value property of plurisubharmonic functions it follows that

$$T_f(r; \omega) \geq c \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \alpha^n = c\nu \int_1^r \frac{dt}{t^{2n-1}} t^{2n} = \frac{c\nu}{2} r^2 + O(1)$$

where ν denote the volume of the unit ball. Therefore

$$\rho_f = \limsup \frac{\log T_f(r)}{\log r} \geq \limsup \frac{\log(r^2)}{\log r} = 2.$$

□

7. NON-KÄHLER MANIFOLDS

Our result is not valid for non-Kähler manifolds. In fact, there are Hopf surfaces X admitting a non-degenerate holomorphic map $f : \mathbb{C}^n \rightarrow X$ of order $\rho_f = 1$ (see [12]). Of course, these Hopf surfaces are non-Kähler and not rationally connected; they do not contain any rational curve at all.

More precisely, in [12] the following is proved:

Theorem 7.1. *Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and let \sim denote the equivalence relation on $\mathbb{C}^2 \setminus \{(0, 0)\}$ given by*

$$v \sim w \iff \exists k : v = \lambda^k w.$$

Let $X = \mathbb{C}^2 \setminus \{(0, 0)\} / \sim$ and let $f : \mathbb{C}^2 \rightarrow X$ be the map induced by $(z, w) \mapsto (z, 1 + zw)$.

Then $\rho_f = 1$.

This result has been generalized by T. Amemiya to the class of Hopf surfaces defined by an equivalence $(z, w) \sim (\lambda^k z, \mu^k w)$ ($k \in \mathbb{Z}$) where λ may be different from μ (but $|\lambda|, |\mu| > 1$).

8. QUESTIONS

The following questions are not expected to have necessarily positive answers.

Let X be an n -dimensional compact Kähler manifold, $f : \mathbb{C}^m \dashrightarrow X$ meromorphic non-degenerate.

1. Is X unirational if $\rho_f = 0$? Is X unirational if $\rho(X) = 0$?

(It should be remarked that $\rho_f = 0$ for every algebraic map, but the condition ρ_f is substantially weaker than algebraicity, as seen by appropriate power series in one variable.)

2. If there exists such an $f : \mathbb{C}^m \dashrightarrow X$, can it be chosen so that $\rho_f < +\infty$? In other words: if there exists an f as above, is $\rho(X) < +\infty$?

3. If $\rho(X) < +\infty$, does there exist some $f : \mathbb{C}^m \dashrightarrow X$ with $\rho_f = \rho(X)$?

4. If X is rationally connected, does there exist a non-degenerate meromorphic map $f : \mathbb{C}^n \dashrightarrow X$? Is then $\rho(X) < +\infty$?

5. If X is RC, and if there exists a non-degenerate $f : \mathbb{C}^m \dashrightarrow X$, is X unirational? (ie: can f be chosen algebraic?). A positive answer would imply that there exists no X with $\rho(X) \in]0, 2[$.

The questions 3 and 4 were raised for $n = 3$ in [4], question 9.5.

6. Is the estimate $\rho(X) \geq 2(1 - \frac{\nu}{n})^{-1}$ in [6] optimal if K_X is pseudoeffective with $\nu(X) =: \nu$? In other words: does there exist X_n with $\nu(X) = \nu$ (or better: with $\kappa(X) = \nu$) and with $\rho(X) \geq 2(1 - \frac{\nu}{n})^{-1}$ for any $n > 0$ and $\nu \in \{0, 1, \dots, n\}$?

REFERENCES

- [1] Amemiya, T.: Orders of meromorphic mappings into Hopf and Inoue surfaces.
- [2] Boucksom, S.; Demailly, J.; Paun, M.; Peternell, T.: The pseudoeffective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Alg. Geometry* **22**, 201–218 (2013)
- [3] Campana, F.: Connexité rationnelle des variétés de Fano. *Ann. Sci. Ec. Norm. Sup* **25**, 539–545 (1992)
- [4] Campana, F.: Orbifolds, special varieties, and classification theory. *Ann. Inst. Fourier* **54**, 499–630 (2004)
- [5] Campana, F.; Demailly, J.P.; Peternell, Th.: Rationally Connected Manifolds and semipositivity of the Ricci Curvature. arXiv:1210.2092
- [6] Campana, F.; Păun, M.: Une généralisation du théorème du Kobayashi-Ochiai. *Manu. math.*, **125**, no. 4, 411–426 (2008)
- [7] Demailly, J.P.: Analytic Methods in Algebraic Geometry. Survey of Modern Mathematics. Vol. I. International Press, Somerville. 2010.
- [8] Griffiths, P.: Entire holomorphic maps in one and several complex variables. *Annals of Math. Studies*, Princeton University Press. 1976

- [9] Graber, T.; Harris, J.; Starr, J.: Rationally connected varieties. *J. Alg.*, **1**, 429–448 (1992)
- [10] Kobayashi, S.; Ochiai, T.: Meromorphic mappings onto compact complex spaces of general type. *Invent. Math.* **31**, 7–16 (1975)
- [11] Kollár, J.; Miyaoka, Y.; Mori, S.: Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.* **36** (1992), no. 3, 765–779.
- [12] J. Noguchi, J. Winkelmann: Order of Meromorphic Maps and rationality of the Image Space. *J. Math. Soc. Japan* **64**, (4), 1169–1180 (2012)
- [13] J. Noguchi, J. Winkelmann: Nevanlinna Theory in Several Complex Variables and Diophantine Approximation. Springer Grundlehren der mathematischen Wissenschaften 350. 416 pages. 2014.

FRÉDÉRIC CAMPANA, INSTITUT ELIE CARTAN BP239, UNIVERSITÉ DE LORRAINE, 54506. VANDOEUVRE-LES-NANCY CEDEX. FRANCE, ET: INSTITUT UNIVERSITAIRE DE FRANCE.

JÖRG WINKELMANN, LEHRSTUHL ANALYSIS II, MATHEMATISCHES INSTITUT, NA 4/73, RUHR-UNIVERSITÄT BOCHUM, 44780 BOCHUM, GERMANY